

ASYMPTOTICALLY FLAT EXTENSIONS OF CMC BARTNIK DATA

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ABSTRACT. Let g be a metric on the 2-sphere \mathbb{S}^2 with positive Gaussian curvature and H be a positive constant. Under suitable conditions on (g, H) , we construct smooth, asymptotically flat 3-manifolds M with non-negative scalar curvature, with outer-minimizing boundary isometric to (\mathbb{S}^2, g) and having mean curvature H , such that near infinity M is isometric to a spatial Schwarzschild manifold whose mass m can be made arbitrarily close to a constant multiple of the Hawking mass of (\mathbb{S}^2, g, H) . Moreover, this constant multiplicative factor depends only on (g, H) and tends to 1 as H tends to 0. The result provides a new upper bound of the Bartnik mass associated to such boundary data.

1. INTRODUCTION

Let g be a metric on the 2-sphere \mathbb{S}^2 for which the first eigenvalue of $-\Delta + K$ is positive, where K is the Gaussian curvature of g . Mantoulidis and Schoen [10] constructed asymptotically flat 3-manifolds with non-negative scalar curvature, whose boundaries are outermost minimal surfaces that are isometric to (\mathbb{S}^2, g) and with ADM mass m_{ADM} [1] that can be made arbitrarily close to the optimal value determined by the Riemannian Penrose inequality [4, 8].

In terms of the quasi-local mass \mathfrak{m}_B introduced by Bartnik [2, 3], the result of Mantoulidis and Schoen can be reformulated as follows: Given smooth *Bartnik data* (Σ, g, H) , i.e., a triple with $\Sigma \simeq \mathbb{S}^2$, g a metric and H a function on Σ , if $H = 0$ and g has positive first eigenvalue $\lambda_1(-\Delta + K) > 0$, then $\mathfrak{m}_B(\Sigma, g, H)$ of such Bartnik data is bounded above by their Hawking mass $\mathfrak{m}_H(\Sigma, g, H)$:

$$(1.1) \quad \mathfrak{m}_B(\Sigma, g, 0) \leq \mathfrak{m}_H(\Sigma, g, 0),$$

where the Hawking mass \mathfrak{m}_H [7] is defined by

$$\mathfrak{m}_H(\Sigma, g, H) := \sqrt{\frac{|\Sigma|_g}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 d\sigma \right).$$

Bartnik's quasi-local mass \mathfrak{m}_B is defined as

$$\mathfrak{m}_B(\Sigma, g, H) := \inf \{ m_{ADM}(M, \gamma) \mid (M, \gamma) \text{ admissible extension of } (\Sigma, g, H) \},$$

where a smooth, asymptotically flat Riemannian 3-manifold (M, γ) with boundary ∂M is called an *admissible extension* of (Σ, g, H) if it has non-negative scalar curvature and if (Σ, g) is isometric to ∂M with mean curvature H . Moreover, it is required

that (M, γ) contains no closed minimal surfaces (except possibly ∂M) or, a fortiori, that ∂M is outer-minimizing in (M, γ) , see [4, 5, 8]. Via the proof of the Riemannian Penrose inequality in [8], the outer-minimizing condition allows one to estimate the Bartnik mass of given Bartnik data from below by its Hawking mass.

For the purpose of this paper, it will make no difference which of the above two conditions – no closed minimal surfaces versus outer-minimizing – is chosen as the extensions we construct satisfy both conditions. Here and in the following, we will abbreviate $\mathbf{m}_H(\Sigma) := \mathbf{m}_H(\Sigma, g, H)$ and $\mathbf{m}_B(\Sigma) := \mathbf{m}_H(\Sigma, g, H)$ whenever the choice of g and H is clear from context.

In the important case of $H = 0$ treated by Mantoulidis and Schoen, combined with the Riemannian Penrose inequality, (1.1) implies $\mathbf{m}_B(\Sigma, g, 0) = \mathbf{m}_H(\Sigma, g, 0)$. We note that the condition $\lambda_1(-\Delta + K) > 0$ used in this setting arises naturally in the context of stable minimal surfaces.

In this paper, we give an analogue of Mantoulidis and Schoen's result for a triple of Bartnik data (Σ, g, H) that is associated to a constant mean curvature (CMC) surface. As a special case of our main Theorem (Theorem 3.1), we have the following result on constructing asymptotically flat extensions with controlled ADM mass.

Theorem 1.1. *Let $(\Sigma \simeq \mathbb{S}^2, g, H)$ be a triple of Bartnik data where $H = H_o$ is a positive constant and g has positive Gaussian curvature. There exist constants $\alpha \geq 0$ and $0 < \beta \leq 1$, depending on g , such that if*

$$(1.2) \quad \frac{1}{16\pi} \int_{\Sigma} H_o^2 d\sigma < \frac{\beta}{1 + \alpha},$$

then for each $m > m_$, with*

$$m_* = \left[1 + \left(\frac{\alpha \left(\frac{1}{16\pi} \int_{\Sigma} H_o^2 d\sigma \right)}{\beta - (1 + \alpha) \left(\frac{1}{16\pi} \int_{\Sigma} H_o^2 d\sigma \right)} \right)^{\frac{1}{2}} \right] \mathbf{m}_H(\Sigma, g, H_o),$$

there exists a smooth, asymptotically flat Riemannian 3-manifold (M, γ) with boundary ∂M and non-negative scalar curvature such that

- (i) ∂M is isometric to (Σ, g) and has constant mean curvature H_o ;
- (ii) M , outside a compact set, is isometric to a spatial Schwarzschild manifold M_m^S of mass m ; and
- (iii) M is foliated by mean convex 2-spheres which eventually coincide with the rotationally symmetric 2-spheres in M_m^S .

We give a few remarks on Theorem 1.1.

Remark 1.1. Similar to the implication of Mantoulidis and Schoen's theorem on the Bartnik mass, a direct corollary of Theorem 1.1 is that

$$(1.3) \quad \mathbf{m}_B(\Sigma, g, H_o) \leq \left[1 + \left(\frac{\alpha \left(\frac{1}{16\pi} \int_{\Sigma} H_o^2 d\sigma \right)}{\beta - (1 + \alpha) \left(\frac{1}{16\pi} \int_{\Sigma} H_o^2 d\sigma \right)} \right)^{\frac{1}{2}} \right] \mathbf{m}_H(\Sigma, g, H_o)$$

for any CMC Bartnik data (Σ, g, H_o) satisfying (1.2). Note that (1.2) implies

$$\frac{1}{16\pi} \int_{\Sigma} H_o^2 d\sigma < \frac{\beta}{1+\alpha} \leq 1,$$

hence $\mathbf{m}_H(\Sigma) > 0$ is implicitly assumed in Theorem 1.1. Moreover, (iii) implies that ∂M is outer-minimizing in M .

Remark 1.2. Inequality (1.3) has the feature that the ratio between the upper bound it provides for $\mathbf{m}_B(\Sigma)$ and the Hawking mass $\mathbf{m}_H(\Sigma)$ tends to 1 as $H_o \rightarrow 0$. However, it assumes (1.2). In [9], Lin and Sormani investigated the Bartnik mass of arbitrary CMC Bartnik data (Σ, g, H_o) , with area $|\Sigma|_g = 4\pi$, and proved that

$$(1.4) \quad \mathbf{m}_B(\Sigma) \leq \mathbf{m}_{aS}(\Sigma) + \mathbf{m}_H(\Sigma),$$

where $\mathbf{m}_{aS}(\Sigma)$, referred to as the *asphericity mass*, is a non-negative constant that is determined only by the metric g on Σ . Note that, in contrast to (1.2), the ratio between the right hand side of (1.4) and the Hawking mass approaches a fixed constant $C > 1$ as $H \rightarrow 0$.

Remark 1.3. The quantity m_* in Theorem 1.1 was first found in [12]. Indeed, the Bartnik mass estimate (1.3) would follow from [12] if one allows admissible extensions to be Lipschitz with distributional non-negative scalar curvature across a hypersurface (cf. [11, 13]). Therefore, our main contribution here is to construct smooth extensions.

We will prove Theorem 1.1 applying the method of Mantoulidis and Schoen [10], which involves two steps. In the first step, one needs a collar extension of (Σ, g, H_o) with positive scalar curvature on $[0, 1] \times \Sigma$ such that the geometric information at $\Sigma_0 := \{0\} \times \Sigma$, corresponding to (Σ, g, H_o) , is suitably propagated to the other end $\Sigma_1 := \{1\} \times \Sigma$, near which the extension is rotationally symmetric. (In the minimal surface case, it is primarily the area of Σ_1 that one wants to compare with Σ_0 . When $H_o > 0$, it is the two Hawking masses that one wants to compare.) In the second step, one smoothly glues the collar extension, at Σ_1 , to a spatial Schwarzschild manifold (suitably deformed in a small region) with mass greater than, but arbitrarily close to, the Hawking mass of Σ_1 .

To implement this process, we make use of the collar extension constructed in [12] in the first step as it provides a good control of the Hawking mass along the collar. For the second step, we prove an elementary result on smoothly gluing a rotationally symmetric manifold with positive scalar curvature and a spatial Schwarzschild manifold with suitably chosen mass (see Proposition 2.1). We give these gluing tools in Section 2, then in Section 3, we prove Theorem 3.1 which implies Theorem 1.1.

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2. GLUING TO A SPATIAL SCHWARZSCHILD MANIFOLD

In this section, we list some tools for gluing together two rotationally symmetric metrics with non-negative scalar curvature, which are used to prove the main result in Section 3. We start with a lemma that is a slight generalization of [10, Lemma 2.2]. As it may be of independent interest, we state it for arbitrary dimension $n \geq 2$.

Lemma 2.1. *Let $f_i : [a_i, b_i] \rightarrow \mathbb{R}^+$, where $i = 1, 2$, be smooth positive functions, and let g_* be the standard metric on \mathbb{S}^n . Suppose that*

- (i) *the metrics $\gamma_i := dt^2 + f_i(t)^2 g_*$ have positive scalar curvature;*
- (ii) *$f_1(b_1) < f_2(a_2)$;*
- (iii) *$1 > f'_1(b_1) > 0$ and $f'_1(b_1) \geq f'_2(a_2) > -1$.*

Then, after suitably translating $[a_i, b_i]$ with $b_1 < a_2$, one can construct a smooth positive function $f : [a_1, b_2] \rightarrow \mathbb{R}^+$ so that:

- (I) *$f \equiv f_1$ on $[a_1, \frac{a_1+b_1}{2}]$, $f \equiv f_2$ on $[\frac{a_2+b_2}{2}, b_2]$, and*
- (II) *$\gamma := dt^2 + f(t)^2 g_*$ has positive scalar curvature on $[a_1, b_2] \times \mathbb{S}^n$.*

Moreover, if $f'_i > 0$ on $[a_i, b_i]$, then f can be constructed such that $f' > 0$ on $[a_1, b_2]$.

Remark 2.1. The assumptions in Lemma 2.1 are weaker than those in [10, Lemma 2.2]. In [10, Lemma 2.2], it was assumed that $f'_i > 0$ and $f''_i > 0$, which implies that $f'_i < 1$ since each γ_i has positive scalar curvature (cf. (2.5)); moreover, we relax the assumption $f'_1(b_1) = f'_2(a_2)$ in [10, Lemma 2.2] to $f'_1(b_1) \geq f'_2(a_2)$, which corresponds to the mean curvature jump condition used in the proof of the positive mass theorem with corners (cf. [11]). This assumption in terms of an inequality helps one simplify the metric gluing procedure used later.

Proof. Since $f'_1(b_1) > 0$, $[a_2, b_2]$ can always be translated so that $b_1 < a_2$ and such that there exists a function $\zeta \in C^1([b_1, a_2])$ satisfying $\zeta(b_1) = f'_1(b_1)$, $\zeta(a_2) = f'_2(a_2)$, $\zeta' \leq 0$, and

$$\int_{b_1}^{a_2} \zeta(t) dt = f_2(a_2) - f_1(b_1) > 0.$$

We now assume that such a translation has been performed (without changing the names of a_2 and b_2). On $[b_1, a_2]$, we define

$$\widehat{f}(t) := f_1(b_1) + \int_{b_1}^t \zeta(x) dx.$$

Then \widehat{f} satisfies

$$\begin{aligned} \widehat{f}(b_1) &= f_1(b_1) \text{ and } \widehat{f}(a_2) = f_2(a_2), \\ \widehat{f}'(b_1) &= f'_1(b_1) \text{ and } \widehat{f}'(a_2) = f'_2(a_2), \\ 1 > f'_1(b_1) &\geq \widehat{f}'(t) \geq f'_2(a_2) > -1 \text{ on } (b_1, a_2), \\ \widehat{f}''(t) &= \zeta'(t) \leq 0 \text{ on } [b_1, a_2]. \end{aligned}$$

On $[a_1, b_2]$, define

$$\tilde{f}(t) := \begin{cases} f_1(t) & \text{on } [a_1, b_1] \\ \hat{f}(t) & \text{on } [b_1, a_2] \\ f_2(t) & \text{on } [a_2, b_2] \end{cases}.$$

Then $\tilde{f} \in C^{1,1}([a_1, b_2])$, C^2 away from b_1, a_2 , and $\tilde{f} > 0$. Slightly abusing notation, we will write \tilde{f}'' on all of $[a_1, b_2]$ extending it to b_1, a_2 by setting $\tilde{f}''(b_1) := f_1''(b_1)$ and $\tilde{f}''(a_2) := f_2''(a_2)$.

We next consider an appropriate mollification of \tilde{f} (cf. [6]). Let $\delta > 0$ be such that

$$\frac{a_1 + b_1}{2} < b_1 - \delta \quad \text{and} \quad a_2 + \delta < \frac{a_2 + b_2}{2}.$$

Let $\eta_\delta : [a_1, b_2] \rightarrow \mathbb{R}_0^+$ be a smooth cutoff function which equals 1 on $[b_1 - \delta, a_2 + \delta]$, vanishes on $[a_1, \frac{a_1+b_1}{2}] \cup [\frac{a_2+b_2}{2}, b_2]$, and satisfies $0 < \eta_\delta(s) < 1$ in the remaining part of the interval. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}_0^+$ be a standard smooth mollifier with compact support in $[-1, 1]$ and $\int_{-\infty}^{\infty} \phi(s) ds = 1$.

For each positive $\varepsilon < \frac{\delta}{4}$, define f_ε by

$$(2.1) \quad f_\varepsilon(t) := \int_{-\infty}^{\infty} \tilde{f}(t - \varepsilon \eta_\delta(t)s) \phi(s) ds, \quad t \in [a_1, b_2]$$

and observe that f_ε is smooth on $[a_1, b_2]$. Then $f_\varepsilon \equiv \tilde{f}$ on $[a_1, \frac{a_1+b_1}{2}] \cup [\frac{a_2+b_2}{2}, b_2]$ and

$$(2.2) \quad f'_\varepsilon(t) = \int_{-\infty}^{\infty} \tilde{f}'(t - \varepsilon \eta_\delta(t)s) (1 - \varepsilon \eta'_\delta(t)s) \phi(s) ds \quad \forall t \in [a_1, b_2].$$

Moreover, since \tilde{f}' is C^0 everywhere and C^1 except at b_1 and a_2 , it can be checked that by standard mollification arguments

$$(2.3) \quad \begin{aligned} f''_\varepsilon(t) &= \frac{d}{dt} \left(\int_{-\infty}^{\infty} \tilde{f}'(t - \varepsilon s) \phi(s) ds, \right) \\ &= \int_{-\infty}^{\infty} \tilde{f}''(s) \phi_\varepsilon(t - s) ds \quad \forall t \in (b_1 - \delta, a_2 + \delta), \end{aligned}$$

where $\phi_\varepsilon(t) := \frac{1}{\varepsilon} \phi(\frac{t}{\varepsilon})$, and

$$(2.4) \quad \begin{aligned} f''_\varepsilon(t) &= \int_{-\infty}^{\infty} \tilde{f}''(t - \varepsilon \eta_\delta(t)s) (1 - \varepsilon \eta'_\delta(t)s)^2 \phi(s) ds \\ &\quad - \varepsilon \int_{-\infty}^{\infty} \tilde{f}'(t - \varepsilon \eta_\delta(t)s) \eta''_\delta(t)s \phi(s) ds, \quad \forall t \notin [b_1 - \frac{1}{4}\delta, a_2 + \frac{1}{4}\delta]. \end{aligned}$$

We claim that, for sufficiently small $\varepsilon > 0$, the metric $dt^2 + f_\varepsilon(t)^2 g_*$ has positive scalar curvature. To show this, recall that given any smooth function $f > 0$, the metric $dt^2 + f(t)^2 g_*$ has positive scalar curvature if and only if

$$(2.5) \quad f''(t) < \frac{n-1}{2f(t)} (1 - f'(t)^2) \quad \forall t.$$

Suggested by (2.5), given any positive $f \in C^1([a_1, b_2])$, we define

$$\Omega[f](t) := \frac{n-1}{2f(t)}(1 - f'(t)^2), \quad t \in [a_1, b_2].$$

Observe that $\Omega[f] \in C^0([a_1, b_2])$. Then, on $[a_1, b_2] \setminus \{b_1, a_2\}$, we have

$$\Omega[\tilde{f}] > \tilde{f}''$$

because $\Omega[\tilde{f}] > 0$ on $[b_1, a_2]$, $\tilde{f}'' \leq 0$ on (b_1, a_2) , and g_i has positive scalar curvature on $[a_i, b_i] \times \mathbb{S}^n$ for $i = 1, 2$. Moreover, for the same reason, we indeed have

$$(2.6) \quad 3d := \inf_{t \in [a_1, b_2] \setminus \{b_1, a_2\}} (\Omega[\tilde{f}] - \tilde{f}'') > 0.$$

Thus,

$$\tilde{f}''(t) \leq \Omega[\tilde{f}](t) - 3d$$

wherever \tilde{f}'' is defined. Now we consider $\Omega[f_\varepsilon]$. By (2.1) and (2.2), $f_\varepsilon \rightarrow \tilde{f}$ in $C^1([a_1, b_2])$ as $\varepsilon \rightarrow 0^+$. Hence, $\Omega[f_\varepsilon] \rightarrow \Omega[\tilde{f}]$ in $C^0([a_1, b_2])$, which shows, for small ε ,

$$\sup_{t \in [a_1, b_2]} |\Omega[\tilde{f}](t) - \Omega[f_\varepsilon](t)| < d.$$

Moreover, by (2.3) and (2.4),

$$f_\varepsilon''(t) < \sup_{|s-t|<\varepsilon} \tilde{f}''(s) + d \quad \forall t \in [a_1, b_2]$$

provided ε is sufficiently small. Therefore, it follows that, for all $t \in [a_1, b_2]$,

$$\begin{aligned} f_\varepsilon''(t) &\leq \sup_{|s-t|<\varepsilon} (\Omega[\tilde{f}](s) - 3d) + d \\ &< \Omega[\tilde{f}](t) - d \\ &< \Omega[f_\varepsilon](t), \end{aligned}$$

where, in the second to last inequality, we also used the fact that $\Omega[\tilde{f}]$ is uniformly continuous on $[a_1, b_2]$, and hence $\Omega[\tilde{f}](s) < \Omega[\tilde{f}](t) + d$ for any s with $|s - t| < \varepsilon$ provided ε is small.

Thus, we have shown that (2.5) holds on $[a_1, b_2]$ with f replaced by f_ε for small ε . Hence, the metric $dt^2 + f_\varepsilon(t)^2 g_*$ has positive scalar curvature and thus $f := f_\varepsilon$ satisfies the conclusions of the theorem for small enough ε . Finally, if it is assumed that $f'_i > 0$, then $\tilde{f}' > 0$ which directly implies $f' = f'_\varepsilon > 0$ as $f_\varepsilon \rightarrow \tilde{f}$ in $C^1([a_1, b_2])$. \square

Remark 2.2. It follows from the above proof that, if $f'_1(b_1) > f'_2(a_2)$, the assumptions $f'_1(b_1) < 1$ and $f'_2(a_2) > -1$ can be weakened to $f'_1(b_1) \leq 1$ and $f'_2(a_2) \geq -1$, respectively. This is because, in this case, one can require $\zeta'(t) < 0$ on $[b_1, a_2]$. Thus, $\tilde{f}'' \leq \max_{[b_1, a_2]} \zeta' < 0$ on (b_1, a_2) while $\Omega[\tilde{f}] \geq 0$ on $[b_1, a_2]$. Therefore, (2.6) still holds.

Proposition 2.1. *Consider a metric $\gamma = ds^2 + f(s)^2 g_*$ on $[a, b] \times \mathbb{S}^2$, where g_* is the standard metric on \mathbb{S}^2 and $f > 0$ is a smooth function on $[a, b]$. Suppose*

- (1) γ has positive scalar curvature;
- (2) $\Sigma_b := \{b\} \times \mathbb{S}^2$ has positive mean curvature; and
- (3) $\mathfrak{m}_H(\Sigma_b) \geq 0$.

Then, for any $m_e > \mathfrak{m}_H(\Sigma_b)$, there exists a smooth, rotationally symmetric, asymptotically flat Riemannian 3-manifold M with boundary ∂M and with non-negative scalar curvature such that

- (i) M , outside a compact set, is isometric to a spatial Schwarzschild manifold of mass m_e ;
- (ii) ∂M has a neighborhood U that is isometric to $([a, \frac{a+b}{2}] \times \mathbb{S}^2, \gamma)$; and
- (iii) if $f' > 0$ on $[a, b]$, then M can be constructed such that every rotationally symmetric sphere in M has positive constant mean curvature.

To prove this proposition, we recall that a spatial Schwarzschild manifold (M_m^S, γ_m) with mass $m > 0$ takes the form of

$$(2.7) \quad (M_m^S, \gamma_m) = \left((2m, \infty) \times \mathbb{S}^2, \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_* \right).$$

Setting $s := \int_{2m}^r (1 - \frac{2m}{r})^{-\frac{1}{2}} dr$, we can slightly extend (M_m^S, γ_m) to

$$(2.8) \quad (M_m^S, \gamma_m) = ([0, \infty) \times \mathbb{S}^2, ds^2 + \bar{u}_m^2(s) g_*),$$

where $\bar{u}_m : [0, \infty) \rightarrow [2m, \infty)$ satisfies $\bar{u}_m(0) = 2m$,

$$(2.9) \quad \bar{u}_m'(s) = \sqrt{1 - \frac{2m}{\bar{u}_m(s)}} \quad \text{and} \quad \bar{u}_m''(s) = \frac{m}{\bar{u}_m(s)^2}.$$

Proof of Proposition 2.1. The Hawking mass of Σ_b is given by

$$(2.10) \quad \mathfrak{m}_H(\Sigma_b) = \frac{f(b)}{2} (1 - f'(b)^2).$$

For simplicity, we set $m_* := \mathfrak{m}_H(\Sigma_b)$, $r_* := f(b)$ and $w_* := f'(b)$, so (2.10) becomes $m_* = \frac{r_*}{2}(1 - w_*^2)$. By assumption, $w_* > 0$ and $m_* \geq 0$. The latter implies $w_* \leq 1$.

Given any $m_e > m_*$, we claim that there exists $s_{m_e} > 0$ such that

$$(2.11) \quad \bar{u}_{m_e}'(s_{m_e}) \leq w_*,$$

$$(2.12) \quad \bar{u}_{m_e}(s_{m_e}) > r_*.$$

In case $m_* > 0$, i.e., $0 < w_* < 1$, this is easily seen by choosing $s_{m_e} > 0$ so that $\bar{u}_{m_e}(s_{m_e}) = w_*$. Then (2.12) follows from (2.9) and the fact that $m > m_*$. If $m_* = 0$, i.e., $w_* = 1$, then (2.11) holds (with a strict inequality) for any $s_{m_e} > 0$. In this case, (2.12) holds for any sufficiently large s_{m_e} .

To proceed, we fix s_{m_e} . Let $\delta \in (0, s_{m_e})$ be a small constant to be chosen later. We now slightly bend open a small piece of the Schwarzschild manifold with mass m_e ,

near $s = s_{m_e}$, so that it has positive scalar curvature and can be smoothly glued here to the collar constructed above using Lemma 2.1.

Following Lemma 2.3 in [10], we consider a smooth function $\sigma : [s_{m_e} - \delta, \infty) \rightarrow \mathbb{R}$ defined by $\sigma(s) := s$ for $s \geq s_{m_e}$, and indirectly via

$$\sigma'(s) := 1 + e^{-1/(s-s_{m_e})^2} \text{ for } s_{m_e} - \delta \leq s \leq s_{m_e}.$$

If δ is small enough then $\sigma(s) > 0$ and the metric

$$\gamma_e := ds^2 + \bar{u}_{m_e}(\sigma(s))^2 g_*$$

has positive scalar curvature on $[s_{m_e} - \delta, s_{m_e}) \times \mathbb{S}^2$. Note that γ_e is identically the Schwarzschild metric with mass m_e on $[s_{m_e}, \infty) \times \mathbb{S}^2$ and $\frac{d}{ds}\bar{u}_{m_e}(\sigma(s)) > 0$, $\forall s \geq s_{m_e} - \delta$.

We now want to compare $\bar{u}_{m_e}(\sigma(s))|_{s=s_{m_e}-\delta}$ and $\frac{d}{ds}\bar{u}_{m_e}(\sigma(s))|_{s=s_{m_e}-\delta}$, with r_* and w_* , respectively. By (2.12), we have

$$(2.13) \quad \bar{u}_{m_e}(\sigma(s))|_{s=s_{m_e}-\delta} > r_*$$

if δ is sufficiently small. By (2.9), one finds

$$\frac{d^2}{ds^2}\bar{u}_{m_e}(\sigma(s)) = m_e \bar{u}_{m_e}(\sigma(s))^{-2} \sigma'(s)^2 + \bar{u}'_{m_e}(\sigma(s)) \sigma''(s).$$

Hence $\frac{d^2}{ds^2}\bar{u}_{m_e}(\sigma(s))|_{s=s_{m_e}} = m_e \bar{u}_{m_e}(s_{m_e})^{-2} > 0$ since $m_e > 0$. Thus, shrinking δ if necessary, we have $\frac{d^2}{ds^2}\bar{u}_{m_e}(\sigma(s)) > 0$ on $[s_{m_e} - \delta, s_{m_e}]$. Combined with (2.11), this implies

$$(2.14) \quad \left. \frac{d}{ds}\bar{u}_{m_e}(\sigma(s)) \right|_{s=s_{m_e}-\delta} < \bar{u}'_{m_e}(s_{m_e}) \leq w_*.$$

Now, by (2.13), (2.14) and as γ_e has positive scalar curvature on $[s_{m_e} - \delta, s_{m_e}) \times \mathbb{S}^2$, we can apply Lemma 2.1 (and Remark 2.2) to smoothly glue γ and γ_e together, with f playing the role of f_1 on $[a_1, b_1] := [a, b]$, and $\bar{u}_{m_e} \circ \sigma$ playing the role of f_2 on $[a_2, b_2] := [s_{m_e} - \delta, s_{m_e} - \frac{\delta}{2}]$. The resulting manifold satisfies all properties we require. \square

3. ASYMPTOTICALLY FLAT EXTENSIONS

Throughout this section, let $(\Sigma \simeq \mathbb{S}^2, g, H)$ be a triple of Bartnik data with $K(g) > 0$ and constant mean curvature $H = H_o > 0$. Let r_o be the area radius of g , i.e., $|\Sigma|_g = 4\pi r_o^2$. We always let $\{g(t)\}_{0 \leq t \leq 1}$ be a smooth path of metrics on Σ such that

- (i) if g is a round metric, $\{g(t)\}_{0 \leq t \leq 1}$ is the constant path with $g(t) = g$, $\forall t \in [0, 1]$; and
- (ii) if g is not a round metric, $\{g(t)\}_{0 \leq t \leq 1}$ is a path of metrics with positive Gaussian curvature satisfying $g(0) = g$, $g(1)$ is a round metric, and

$$\text{tr}_{g(t)} g'(t) = 0, \quad \forall t \in [0, 1].$$

(Existence of such a $\{g(t)\}_{0 \leq t \leq 1}$ is given by Mantoulidis-Schoen's proof of [10, Lemma 2.1].)

As in [12, Section 2], we let α and β be two constants determined by such a path via

$$(3.1) \quad \begin{aligned} \alpha &:= \frac{1}{4} \max_{t \in [0,1]} \max_{\Sigma} |g'(t)|_{g(t)}^2, \\ \beta &:= r_o^2 \min_{t \in [0,1]} \min_{\Sigma} K(g(t)). \end{aligned}$$

Clearly,

- (a) $\beta = 1$ and $\alpha = 0$, if g is a round metric; and
- (b) $0 < \beta < 1$ and $\alpha > 0$, if g is not a round metric.

In assertion (b), one uses the Gauß-Bonnet Theorem and the fact $|\Sigma|_{g(t)} = 4\pi r_o^2$, $\forall t$.

The following theorem illustrates how the gluing tools in the previous section and the collar extension in [12] are combined to produce asymptotically flat extension with suitable CMC Bartnik data.

Theorem 3.1. *Let $(\Sigma \simeq \mathbb{S}^2, g, H)$ be a triple of Bartnik data where g has positive Gaussian curvature and $H = H_o$ is a positive constant. Let r_o be the area radius of g , i.e., $|\Sigma|_g = 4\pi r_o^2$. Let $\{g(t)\}_{0 \leq t \leq 1}$ be a path of metrics given in (i) or (ii) above. Let $\alpha \geq 0$ and $0 < \beta \leq 1$ be the constants defined in (3.1) for this path $\{g(t)\}_{0 \leq t \leq 1}$. Suppose the condition*

$$(3.2) \quad \frac{1}{4} H_o^2 r_o^2 < \frac{\beta}{1 + \alpha}$$

holds. Given any $m \in (-\infty, \frac{1}{2} r_o)$ satisfying

$$(3.3) \quad \frac{1}{4} H_o^2 r_o^2 < \frac{\beta}{1 + \alpha} \left(1 - \frac{2m}{r_o}\right),$$

let $k > 0$ be the constant given by

$$(3.4) \quad k := \frac{H_o r_o}{2} \left(1 - \frac{2m}{r_o}\right)^{-\frac{1}{2}}.$$

Define

$$(3.5) \quad m_* := \begin{cases} \frac{1}{2} r_o \left[\frac{\frac{1}{4} H_o^2 r_o^2 \alpha}{(\beta - \frac{1}{4} H_o^2 r_o^2 \alpha) - k^2} \right]^{\frac{1}{2}} (1 - k^2) + \mathbf{m}_H(\Sigma), & \text{if } m < 0, \\ \frac{1}{2} r_o \left[\frac{\alpha k^2}{\beta - (1 + \alpha) k^2} \right]^{\frac{1}{2}} (1 - k^2) + \mathbf{m}_H(\Sigma), & \text{if } m \geq 0, \end{cases}$$

where $\mathbf{m}_H(\Sigma) := \mathbf{m}_H(\Sigma, g, H_o)$. Then, for any $m_e > m_*$, there exists a smooth, asymptotically flat Riemannian 3-manifold (M, γ) with boundary ∂M and non-negative scalar curvature such that

- (i) ∂M is isometric to (Σ, g) and has constant mean curvature H_o ;

- (ii) M , outside a compact set, is isometric to a spatial Schwarzschild manifold $M_{m_e}^S$ of mass m_e ; and
- (iii) M is foliated by mean convex 2-spheres which eventually coincide with the rotationally symmetric 2-spheres in $M_{m_e}^S$.

Proof. We note that (3.2) implies

$$\frac{1}{16\pi} \int_{\Sigma} H_o^2 d\sigma < \frac{\beta}{1+\alpha} \leq 1;$$

that is $\mathbf{m}_H(\Sigma) > 0$ by hypothesis.

We first consider the case that g is not a round metric, i.e. $\alpha > 0$. In this case, we will first prove the theorem under the additional assumption that

$$(3.6) \quad g(t) = g(1), \quad \forall t \in [1 - \theta, 1]$$

for some $\theta \in (0, \frac{1}{3})$. Such a condition is imposed so that later we can directly apply Proposition 2.1.

Now we describe the collar extension produced in [12, Proposition 2.1]. Given the constant $m \in (-\infty, \frac{1}{2}r_o)$, consider part of the spatial Schwarzschild metric

$$\gamma_m = \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_*$$

defined on $[r_o, \infty) \times \mathbb{S}^2$. By a change of variable $s := \int_{r_o}^r \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} dr$, one can write

$$\gamma_m = ds^2 + u_m^2(s) g_*,$$

where $s \in [0, \infty)$ and u_m is the inverse function of $s = s(r)$. (Note that, in the case of $m \geq 0$, u_m is related to \bar{u}_m in (2.8) by $u_m(t) := \bar{u}_m(s_0 + t)$, where $s_0 > 0$ is given by $\bar{u}_m(s_0) = r_o$.) Next, by (3.2) – (3.4), we have

$$(3.7) \quad \begin{cases} \beta - \left[1 + \left(1 - \frac{2m}{r_o}\right)\alpha\right] k^2 > 0, & \text{if } m < 0 \\ \beta - (1 + \alpha)k^2 > 0, & \text{if } m \geq 0. \end{cases}$$

Therefore, we can define a constant $A_o > 0$ by

$$A_o := \begin{cases} r_o \left[\frac{\alpha}{\beta - \left[1 + \left(1 - \frac{2m}{r_o}\right)\alpha\right] k^2} \right]^{\frac{1}{2}}, & \text{if } m < 0 \\ r_o \left[\frac{\alpha}{\beta - (1 + \alpha)k^2} \right]^{\frac{1}{2}}, & \text{if } m \geq 0. \end{cases}$$

Applying Proposition 2.1 of [12] (and the subsequent Remark 2.1), we know the metric

$$\gamma := A_o^2 dt^2 + \frac{u_m(A_o k t)^2}{r_o^2} g(t)$$

defined on $N := [0, 1] \times \Sigma$ satisfies:

- (i) $R(\gamma) > 0$, where $R(\gamma)$ is the scalar curvature of γ ;
- (ii) the induced metric on $\Sigma_0 := \{0\} \times \Sigma$ by γ is g ;
- (iii) the mean curvature $H(0)$ of Σ_0 is $H(0) = H_o$;
- (iv) $\Sigma_t := \{t\} \times \mathbb{S}^2$ has positive constant mean curvature $\forall t \in [0, 1]$; and
- (v) the Hawking mass of $\Sigma_1 = \{1\} \times \Sigma$ is given by

$$\mathbf{m}_H(\Sigma_1) = \frac{1}{2}[u_m(A_o k) - r_o](1 - k^2) + \mathbf{m}_H(\Sigma).$$

Moreover, as in [12, Section 3], one can estimate $\mathbf{m}_H(\Sigma_1)$ by

$$(3.8) \quad \mathbf{m}_H(\Sigma_1) \leq \begin{cases} \frac{1}{4}H_o r_o A_o (1 - k^2) + \mathbf{m}_H(\Sigma) & \text{if } m < 0 \\ \frac{1}{2}A_o k (1 - k^2) + \mathbf{m}_H(\Sigma) & \text{if } m \geq 0. \end{cases}$$

Here

$$\frac{1}{4}H_o r_o A_o = \frac{1}{2}r_o \left[\frac{\frac{1}{4}H_o^2 r_o^2 \alpha}{(\beta - \frac{1}{4}H_o^2 r_o^2 \alpha) - k^2} \right]^{\frac{1}{2}} \text{ if } m < 0,$$

and

$$\frac{1}{2}A_o k = \frac{1}{2}r_o \left[\frac{\alpha k^2}{\beta - (1 + \alpha)k^2} \right]^{\frac{1}{2}} \text{ if } m \geq 0.$$

In other words, (3.8) is simply to assert

$$\mathbf{m}_H(\Sigma_1) \leq m_*.$$

By (3.3) and (3.4), we have $k^2 < \frac{\beta}{1+\alpha} < 1$. Therefore,

$$0 < \mathbf{m}_H(\Sigma) < \mathbf{m}_H(\Sigma_1) \leq m_*.$$

Upon a change of variable $s := A_o t$, (N, γ) becomes

$$\left([0, A_o] \times \Sigma, ds^2 + \frac{f(s)^2}{r_o^2} g(A_o^{-1} s) \right),$$

where $f(s) = u_m(ks)$ and $f'(s) > 0$. On $[(1 - \theta)A_o, A_o] \times \Sigma$, by (3.6),

$$\gamma = ds^2 + f(s)^2 g_*$$

for some fixed round metric g_* of area 4π . The theorem now follows readily from Proposition 2.1.

Next, we show that the theorem still holds when condition (3.6) is removed. The idea is to simply approximate $\{g(t)\}_{0 \leq t \leq 1}$ by a path satisfying (3.6). Given any small $\theta \in (0, \frac{1}{3})$, let $\xi_\theta(t) \geq 0$ be a smooth function on $[0, 1]$ such that

$$(3.9) \quad \begin{cases} \xi_\theta(t) = \frac{t}{1 - 2\theta}, \quad \forall t \in [0, 1 - 3\theta], \\ \xi_\theta(t) = 1, \quad \forall t \in [1 - \theta, 1], \\ 0 \leq \xi'_\theta(t) \leq \frac{1}{1 - 2\theta} \quad \forall t \in [0, 1]. \end{cases}$$

(For instance, such an $\xi_\theta(t)$ can be obtained by a usual mollification of the piecewise smooth function that equals $\frac{t}{1-2\theta}$ on $[0, 1-2\theta]$ and equals 1 on $[1-2\theta, 1]$.) Consider the reparametrized path

$$g_\theta(t) := g(\xi_\theta(t)),$$

which satisfies $g_\theta(t) = g(1)$ for $t \in [1-\theta, 1]$. Let α_θ and β_θ be the corresponding constants defined in (3.1) with $g(t)$ replaced by $g_\theta(t)$. Clearly,

$$(3.10) \quad \beta_\theta = \beta.$$

We claim

$$(3.11) \quad \lim_{\theta \rightarrow 0} \alpha_\theta = \alpha.$$

To see this, note that

$$|g'_\theta(t)|_{g_\theta(t)}^2 = (\xi'_\theta(t))^2 |g'(\xi_\theta(t))|_{g(\xi_\theta(t))}^2, \quad \forall t \in [0, 1].$$

Hence, by (3.9),

$$|g'_\theta(t)|_{g_\theta(t)}^2 = (1-2\theta)^{-2} |g'(\xi_\theta(t))|_{g(\xi_\theta(t))}^2, \quad \forall t \in [0, 1-3\theta],$$

which shows

$$(3.12) \quad \alpha_\theta \geq (1-2\theta)^{-2} \frac{1}{4} \max_{s \in [0, \frac{1-3\theta}{1-2\theta}]} \max_{\Sigma} |g'(s)|_{g(s)}^2.$$

On the other hand, also by (3.9),

$$(3.13) \quad \alpha_\theta \leq (1-2\theta)^{-2} \alpha.$$

Letting $\theta \rightarrow 0$, (3.11) follows from (3.12) and (3.13).

Now, by (3.10) and (3.11), we may assume

$$\frac{1}{4} H_o^2 r_o^2 < \frac{\beta_\theta}{1 + \alpha_\theta} \quad \text{and} \quad \frac{1}{4} H_o^2 r_o^2 < \frac{\beta_\theta}{1 + \alpha_\theta} \left(1 - \frac{2m}{r_o}\right)$$

for sufficiently small θ . For these θ , Theorem 3.1 holds for every m_e satisfying

$$(3.14) \quad m_e > m_{\theta*},$$

where $m_{\theta*}$ is the constant in (3.5) defined via α_θ and β_θ . Now suppose $m_e > m_*$. Since $\lim_{\theta \rightarrow 0} m_{\theta*} = m_*$ by (3.10) and (3.11), m_e must satisfy (3.14) for small θ . Thus, the conclusion of Theorem 3.1 holds for m_e . This completes the proof of Theorem 3.1 if g is not a round metric.

Finally, we consider the case that g is a round metric, say $g = r_o^2 g_*$. In this case, we have $\alpha = 0$, $m_* = \mathbf{m}_H(\Sigma)$, and (Σ, g, H_o) is simply the boundary data of the spatial Schwarzschild metric $\gamma_{m_*} = ds^2 + \bar{u}_{m_*}(s)^2 g_*$ on $[s_0, \infty) \times \mathbb{S}^2$, where $\bar{u}_{m_*}(s_0) = r_o$. Now pick any $s_1 > s_0$. Let $\delta > 0$ be a small constant to be chosen. Similar to Lemma 2.3 in [10], we consider a smooth function τ on $[s_0, s_1 + \delta]$ defined by $\tau(s) = s$ for $s \in [s_0, s_1]$, and indirectly via

$$\tau'(s) = 1 - e^{-1/(s-s_1)^2}, \quad s_1 \leq s \leq s_1 + \delta.$$

Define a metric $\gamma_B := ds^2 + \bar{u}_{m_*}(\tau(s))^2 g_*$. Then γ_B coincides with γ_{m_*} on $[s_0, s_1] \times \mathbb{S}^2$ and has positive scalar curvature on $(s_1, s_1 + \delta] \times \mathbb{S}^2$ for sufficiently small δ . Let $\mathbf{m}_H(\Sigma_{s_1})$ and $\mathbf{m}_H(\Sigma_{s_1 + \delta})$ denote the Hawking masses of $\Sigma_{s_1} = \{s_1\} \times \mathbb{S}^2$ and $\Sigma_{s_1 + \delta} = \{s_1 + \delta\} \times \mathbb{S}^2$ with respect to γ_B , respectively. Then $\mathbf{m}_H(\Sigma_{s_1}) = m_*$, and $\mathbf{m}_H(\Sigma_{s_1 + \delta})$ can be made arbitrarily close to $\mathbf{m}_H(\Sigma_{s_1})$, provided δ is sufficiently small. Thus, given any fixed $m_e > m_*$, we can choose δ sufficiently small to ensure

$$m_e > \mathbf{m}_H(\Sigma_{s_1 + \delta}) > 0.$$

Moreover, we can assume $\frac{d}{ds}\bar{u}_{m_*}(\tau(s))|_{s=s_1 + \delta} > 0$, i.e., $\Sigma_{s_1 + \delta}$ has positive mean curvature with respect to γ_B . The theorem now again follows from Proposition 2.1. \square

Remark 3.1. Taking $m = 0$ in Theorem 3.1, one has $k = \frac{1}{2}H_o r_o$ and

$$\frac{1}{2}A_o k(1 - k^2) = \left[\frac{\alpha \left(\frac{1}{4}H_o^2 r_o^2\right)}{\beta - (1 + \alpha) \left(\frac{1}{4}H_o^2 r_o^2\right)} \right]^{\frac{1}{2}} \mathbf{m}_H(\Sigma_o).$$

Hence, Theorem 3.1 implies Theorem 1.1.

Remark 3.2. In Theorem 3.1, if g tends to a fixed round metric on Σ in the $C^{2,\eta}$ -norm for some $\eta \in (0, 1)$, then, by the proof of [12, Proposition 4.1], one can find a particular path $\{g(t)\}_{0 \leq t \leq 1}$ such that the constants α and β satisfy $\alpha \rightarrow 0$ and $\beta \rightarrow 1$.

Remark 3.3. Varying m subject to (3.3), Theorem 3.1 gives different estimates m_* for the Bartnik mass $\mathbf{m}_B(\Sigma)$. We refer readers to Appendix A of [12] for this optimality analysis given in a different context.

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